

Structure of conformal metrics on \mathbb{R}^n with constant Q-curvature

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Abstract

In this article we study the nonlocal equation

$$(-\Delta)^{\frac{n}{2}}u = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{nu} dx < \infty,$$

which arises in the conformal geometry. Inspired by the previous work of C. S. Lin and L. Martinazzi in even dimension and T. Jin, A. Maalaoui, L. Martinazzi, J. Xiong in dimension three we classify all solutions to the above equation in terms of their behavior at infinity.

1 Introduction to the problem and the main theorems

In this paper we consider the equation

$$(-\Delta)^{\frac{n}{2}}u = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n. \quad (1)$$

Here we assume that

$$V := \int_{\mathbb{R}^n} e^{nu} dx < \infty, \quad (2)$$

and we shall see both the left and right-hand side of (1) as tempered distributions. In order to define the left-hand side of (1) as a tempered distribution, one possibility is to follow the approach of [14], i.e. we see the operator $(-\Delta)^{\frac{n}{2}}$ as $(-\Delta)^{\frac{n}{2}} := (-\Delta)^{\frac{1}{2}} \circ (-\Delta)^{\frac{n-1}{2}}$ for $n \geq 1$ odd integer with the convention that $(-\Delta)^0$ is the identity, where $(-\Delta)^{\frac{1}{2}}$ is defined as follows. First for $s > 0$ consider the space

$$L_s(\mathbb{R}^n) := \left\{ v \in L_{loc}^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}} dx < \infty \right\}. \quad (3)$$

Then for $v \in L_s(\mathbb{R}^n)$ we define $(-\Delta)^s v$ as the tempered distribution defined by

$$\langle (-\Delta)^s v, \varphi \rangle := \int_{\mathbb{R}^n} v (-\Delta)^s \varphi dx \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n), \quad (4)$$

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where

$$\mathcal{S}(\mathbb{R}^n) := \left\{ u \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x|^N |D^\alpha u(x)| < \infty \text{ for all } N \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^n \right\}$$

is the Schwartz space, and

$$(-\Delta)^s \varphi(\xi) = |\xi|^{2s} \hat{\varphi}(\xi), \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Here the normalized Fourier transform is defined by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx, \quad f \in L^1(\mathbb{R}^n).$$

Notice that the integral in (4) converges thanks to Proposition 2.1 below.

Then a possible definition of the equation

$$(-\Delta)^{\frac{n}{2}} u = f \quad \text{in } \mathbb{R}^n \tag{5}$$

is the following:

Definition 1.1 *Given $f \in \mathcal{S}'(\mathbb{R}^n)$, we say that u is a solution of (5) if*

$$u \in W_{loc}^{n-1,1}(\mathbb{R}^n), \quad \Delta^{\frac{n-1}{2}} u \in L_{\frac{1}{2}}(\mathbb{R}^n),$$

and

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} u(x) (-\Delta)^{\frac{1}{2}} \varphi(x) dx = \langle f, \varphi \rangle, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{6}$$

While Definition 1.1 is general enough for our purposes, requiring a priori that a solution to (1) belongs to $W_{loc}^{n-1,1}(\mathbb{R}^n)$ might sound unnecessarily restrictive. In fact it is possible to relax Definition 1.1 as follows.

Definition 1.2 *Given $f \in \mathcal{S}'(\mathbb{R}^n)$, a function $u \in L_{\frac{n}{2}}(\mathbb{R}^n)$ is a solution of (5) if*

$$\int_{\mathbb{R}^n} u(x) (-\Delta)^{\frac{n}{2}} \varphi(x) dx = \langle f, \varphi \rangle, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{7}$$

Notice again that the integral in (6) and (7) are converging by Proposition 2.1 below.

As we shall see, a function u solving (1)-(2) in the sense of Definition 1.2 also solves (1) in the sense of Definition 1.1, and conversely, see Proposition 2.6 below. Therefore, from now on a solution of (1)-(2) will be intended in the sense of Definition 1.1. In fact it turns out that such solutions enjoy even more regularity:

Theorem 1.1 *Let u be a solution of (1)-(2) (in the sense of Definition 1.1 or 1.2). Then u is smooth.*

Geometrically any solution u of (1)-(2) corresponds to a conformal metric $g_u := e^{2u}|dx|^2$ on \mathbb{R}^n ($|dx|^2$ is the Euclidean metric on \mathbb{R}^n) such that the Q -curvature of g_u is constant $(n-1)!$. Moreover the volume and the total Q -curvature of the metric g_u are $V = \int_{\mathbb{R}^n} e^{nu} dx < \infty$ and $\int_{\mathbb{R}^n} (n-1)!e^{nu} dx < \infty$ respectively. When $n = 1$ a geometric interpretation of (1) in terms of holomorphic immersion of $\overline{D^2}$ into \mathbb{C} was given in [[7], Theorem 1.3]. If u is a solution of (1) then for any constant c , $\tilde{u} := u - c$ satisfies

$$(-\Delta)^{\frac{n}{2}} \tilde{u} = (n-1)!e^{nc}e^{n\tilde{u}} \quad \text{in } \mathbb{R}^n.$$

This shows that we could take any arbitrary positive constant instead of $(n-1)!$ in (1), but we restrict ourselves to the fixed constant $(n-1)!$ because it is the constant Q -curvature of the round sphere S^n .

Now we shall address the following question: What are the solutions to (1) and in particular how do they behave at infinity?

It is well known that the equation (1) possess the following explicit solution

$$u(x) = \log \left(\frac{2}{1 + |x|^2} \right),$$

obtained by pulling back the round metric on S^n via the stereographic projection.

By translating and rescaling this function u one can produce a class of solutions, namely

$$u_{\lambda, x_0}(x) := \log \left(\frac{2\lambda}{1 + \lambda^2|x - x_0|^2} \right),$$

for every $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. Any such u_{λ, x_0} is called spherical solution. W. Chen-C. Li [6] showed that these are the only solutions in dimension two but in higher dimension nonspherical solutions do exist as shown by A. Chang-W. Chen [4]. C. S. Lin [16] for $n = 4$ and L. Martinazzi [17] for $n \geq 4$ even classified all solutions of (1)-(2) and they proved:

Theorem A ([16], [17]) *Any solution u of (1)-(2) with n even has the asymptotic behavior*

$$u(x) = -P(x) - \alpha \log |x| + o(\log |x|) \quad (8)$$

where $\alpha = \frac{2V}{|S^n|}$, $\frac{o(\log |x|)}{\log |x|} \rightarrow 0$ as $|x| \rightarrow \infty$ and P is a polynomial bounded from below and of degree at most $n - 2$.

A partial converse of Theorem A holds true. For a given $0 < \alpha < 2$ and a given polynomial P such that $\deg(P) \leq n - 2$ and $x \cdot \nabla P(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, J. Wei-D. Ye [21] in dimension four and A. Hyder-L. Martinazzi [13] in even dimension $n \geq 4$ proved the existence of solutions of (1)-(2) with asymptotic behavior given in (8).

When n is odd things are more complex as the operator $(-\Delta)^{\frac{n}{2}}$ is nonlocal. In a recent work T. Jin, A. Maalaoui, L. Martinazzi, J. Xiong have proven the following theorem in dimension three:

Theorem B ([14]) *Let u be a smooth solution of (1)-(2) with $n = 3$. Then u has the asymptotic behavior given by (8), where P is a polynomial of degree 0 or 2 bounded from below, $\alpha \in (0, 2]$ and $\alpha = 2$ if and only if $\deg(P) = 0$. Moreover for every $0 < \alpha < 2$ there exist at least one smooth solution of (1)-(2).*

In analogy with Theorem A and B we study the asymptotic behavior of smooth solutions to the problem (1)-(2) in odd dimension. In order to do that we define

$$v(x) := \frac{(n-1)!}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1+|y|}{|x-y|} \right) e^{nu(y)} dy, \quad \gamma_n = \frac{(n-1)!}{2} |S^n|, \quad (9)$$

where u is a smooth solution of (1)-(2) and we prove

Theorem 1.2 *Let $n \geq 3$ be any odd integer and let u be a smooth solution of (1)-(2). Then*

$$u = v + P,$$

where P is a polynomial of degree at most $n-1$ bounded from above, v is given by (9) and it satisfies

$$v(x) = -\alpha \log |x| + o(\log |x|), \quad \text{as } |x| \rightarrow \infty,$$

where $\alpha = \frac{2V}{|S^n|}$. Moreover

$$\lim_{|x| \rightarrow \infty} D^\beta v(x) = 0 \text{ for every multi-index } \beta \in \mathbb{N}^n \text{ with } 0 < |\beta| \leq n-1.$$

Under certain assumptions on the polynomial P , a partial converse of Theorem 1.2 has been proven by A. Hyder [12], namely

Theorem C ([12]) *Let $n \geq 3$ be an odd integer. For any given $V \in (0, |S^n|)$ and any given polynomial P of degree at most $n-1$ such that*

$$P(x) \rightarrow \infty \quad \text{as } |x| \rightarrow \infty, \quad (10)$$

there exists $u \in C^\infty(\mathbb{R}^n) \cap L_{\frac{n}{2}}(\mathbb{R}^n)$ solution of (1)-(2) having the asymptotic behavior given in (8) with $\alpha = \frac{2V}{|S^n|}$.

Using Theorem 1.2 one can obtain necessary and sufficient conditions under which any solution of (1)-(2) is spherical. More precisely we have the following theorem.

Theorem 1.3 *Let u be a smooth solution of (1)-(2). Then the following are equivalent:*

- (i) u is a spherical solution.
- (ii) $\deg(P) = 0$, where P is the polynomial given by Theorem 1.2.
- (iii) $u(x) = o(|x|^2)$ as $|x| \rightarrow \infty$.
- (iv) $\lim_{|x| \rightarrow \infty} \Delta^j u(x) = 0$ for $j = 1, 2, \dots, \frac{n-1}{2}$.
- (v) $\liminf_{|x| \rightarrow \infty} R_{g_u} > -\infty$, where R_{g_u} is the scalar curvature of g_u .
- (vi) $\pi^* g_u$ can be extended to a Riemannian metric on S^n , where π is the stereographic projection.

Moreover, if u is not a spherical solution then there exists a j with $1 \leq j \leq \frac{n-1}{2}$ and a constant $c < 0$ such that

$$\lim_{|x| \rightarrow \infty} \Delta^j u(x) = c. \quad (11)$$

The equivalence (i) \Leftrightarrow (vi) was proven by Chang-Yang [5] for $n \geq 3$ odd or even using moving plane technique.

In dimension 3 and 4 if u is a smooth solution of (1)-(2) then $V \in (0, |S^n|]$ (see [16], [14]) but V could be greater than $|S^n|$ in higher dimension. For instance in dimension 6, L. Martinazzi [18] proved the existence of solution with large volume. In a recent work X. Huang-D. Ye [11] in dimension $n = 4k+2$ with $k \geq 1$ have shown the existence of solution for any volume $V \in (0, \infty)$. What would be the precise range of the volume V in dimension $n \geq 5$ odd or n is of the form $n = 4k$ and $k \geq 2$ is an open question.

We also mention that using different techniques F. Da Lio, L. Martinazzi and T. Riviere [7] have discussed the case in one dimension, proving that all solutions are spherical.

2 Definitions, regularity issues and proof of Theorem 1.1

Proposition 2.1 *For any $s > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$|(-\Delta)^s \varphi(x)| \leq \frac{C}{|x|^{n+2s}},$$

where $(-\Delta)^s \varphi := (-\Delta)^\sigma \circ (-\Delta)^k \varphi$, where $\sigma \in [0, 1)$, $k \in \mathbb{N}$ and $s = k + \sigma$.

In order to prove Proposition 2.1 let us introduce the spaces

$$\begin{aligned} \mathcal{S}_k(\mathbb{R}^n) &:= \{\varphi \in \mathcal{S}(\mathbb{R}^n) : D^\alpha \hat{\varphi}(0) = 0, \text{ for } |\alpha| \leq k\} \\ &= \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} y^\alpha \varphi(y) dy = 0, \text{ for } |\alpha| \leq k \right\}, \quad k = 0, 1, 2, \dots \\ \mathcal{S}_{-1}(\mathbb{R}^n) &:= \mathcal{S}(\mathbb{R}^n) \end{aligned}$$

Proposition 2.1 easily follows from the remark that $\Delta^k \varphi \in \mathcal{S}_{2k-1}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and from Lemma 2.2 below.

Lemma 2.2 *Let $\varphi \in \mathcal{S}_k(\mathbb{R}^n)$ and $\sigma \in (0, 1)$. Then*

$$|(-\Delta)^\sigma \varphi(x)| \leq \frac{C}{|x|^{n+2\sigma+k+1}}, \quad x \in \mathbb{R}^n.$$

Proof. Since $(-\Delta)^\sigma \varphi \in C^\infty(\mathbb{R}^n)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it suffices to prove the lemma for large x . For a fix $x \in \mathbb{R}^n$ we split \mathbb{R}^n into

$$A_1 := B_{\frac{|x|}{2}} \quad \text{and} \quad A_2 := \mathbb{R}^n \setminus B_{\frac{|x|}{2}}.$$

Then using (28) we have

$$|(-\Delta)^\sigma \varphi(x)| \leq \frac{1}{2} C_{n,\sigma} (I_1 + I_2),$$

where

$$I_i := \left| \int_{A_i} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(x)}{|y|^{n+2\sigma}} dy \right| \quad i = 1, 2.$$

Noticing that on A_1

$$|\varphi(x+y) + \varphi(x-y) - 2\varphi(x)| \leq \|D^2\varphi\|_{L^\infty(B_{\frac{|x|}{2}}(x))} |y|^2,$$

we get

$$I_1 \leq \|D^2\varphi\|_{L^\infty(B_{\frac{|x|}{2}}(x))} \int_{A_1} \frac{dy}{|y|^{n-2+2\sigma}} \leq C \|D^2\varphi\|_{L^\infty(B_{\frac{|x|}{2}}(x))} |x|^{2-2\sigma}.$$

On the other hand

$$\begin{aligned} I_2 &\leq 2|\varphi(x)| \int_{A_2} \frac{dy}{|y|^{n+2\sigma}} + 2 \left| \int_{A_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}} dy \right| \leq 2 \left| \int_{A_2} \frac{\varphi(x-y)}{|y|^{n+2\sigma}} dy \right| + C|\varphi(x)||x|^{-2\sigma} \\ &=: 2I_3 + C|\varphi(x)||x|^{-2\sigma}. \end{aligned}$$

Changing the variable $y \mapsto x-y$ we have

$$\begin{aligned} I_3 &= \left| \int_{|x-y| > \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| \leq \left| \int_{|x-y| > \frac{|x|}{2}, |y| > \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| + \left| \int_{|y| < \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| \\ &\leq \left| \int_{|y| < \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \right| + C\|\varphi\|_{L^\infty(A_2)} |x|^{-2\sigma} \\ &=: I_4 + C\|\varphi\|_{L^\infty(A_2)} |x|^{-2\sigma}. \end{aligned}$$

Finally, to bound I_4 we use the fact that $\varphi \in S_k$. Setting $f(x) = \frac{1}{|x|^{n+2\sigma}}$ and using

$$\sum_{|\alpha| \leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{\mathbb{R}^n} y^\alpha \varphi(y) dy = 0, \quad x \neq 0,$$

we obtain

$$\begin{aligned} &\int_{|y| < \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy \\ &= \int_{|y| < \frac{|x|}{2}} \frac{\varphi(y)}{|x-y|^{n+2\sigma}} dy - \sum_{|\alpha| \leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{|y| < \frac{|x|}{2}} y^\alpha \varphi(y) dy - \sum_{|\alpha| \leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{|y| > \frac{|x|}{2}} y^\alpha \varphi(y) dy \\ &= \int_{|y| < \frac{|x|}{2}} \varphi(y) \left(f(x-y) - \sum_{|\alpha| \leq k} y^\alpha \frac{D^\alpha f(x)}{\alpha!} \right) dy - \sum_{|\alpha| \leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{|y| > \frac{|x|}{2}} y^\alpha \varphi(y) dy \\ &= \int_{|y| < \frac{|x|}{2}} \varphi(y) \sum_{|\beta| = k+1} y^\beta R_\beta(\xi_y) dy - \sum_{|\alpha| \leq k} \frac{D^\alpha f(x)}{\alpha!} \int_{|y| > \frac{|x|}{2}} y^\alpha \varphi(y) dy, \end{aligned}$$

where $R_\beta(\xi_y)$ satisfies

$$f(x-y) = \sum_{|\alpha| \leq k} y^\alpha \frac{D^\alpha f(x)}{\alpha!} + \sum_{|\beta| = k+1} y^\beta R_\beta(\xi_y), \quad |y| < \frac{|x|}{2}, \quad \xi_y \in B_{\frac{|x|}{2}}(x),$$

and

$$|R_\beta(\xi_y)| \leq C \max_{|\alpha|=k+1} \max_{z \in B_{\frac{|x|}{2}}(x)} |D^\alpha f(z)| \leq \frac{C}{|x|^{n+2\sigma+k+1}}.$$

Therefore,

$$\begin{aligned} I_4 &\leq \sum_{|\beta|=k+1} \int_{|y| < \frac{|x|}{2}} |\varphi(y)| |y|^{|\beta|} |R_\beta(\xi_y)| dy + \sum_{|\alpha| \leq k} \frac{|D^\alpha f(x)|}{\alpha!} \int_{A_2} |y|^{|\alpha|} |\varphi(y)| dy \\ &\leq \frac{C}{|x|^{n+2\sigma+k+1}} \int_{\mathbb{R}^n} |\varphi(y)| |y|^{k+1} dy + \|\sqrt{|\varphi|}\|_{L^\infty(A_2)} \sum_{|\alpha| \leq k} \frac{|D^\alpha f(x)|}{\alpha!} \int_{\mathbb{R}^n} |y|^{|\alpha|} \sqrt{|\varphi(y)|} dy, \end{aligned}$$

and complete the proof. \square

Lemma 2.3 *Let $f \in L^1(\mathbb{R}^n)$. We set*

$$\tilde{v}(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1+|y|}{|x-y|} \right) f(y) dy, \quad x \in \mathbb{R}^n. \quad (12)$$

Then

(i) $\tilde{v} \in W_{loc}^{n-1,1}(\mathbb{R}^n)$ and

$$D^\alpha \tilde{v} = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} D_x^\alpha \log \left(\frac{1+|y|}{|x-y|} \right) f(y) dy, \quad 0 \leq |\alpha| \leq n-1.$$

(ii) $D^\alpha \tilde{v} \in L_{\frac{1}{2}}(\mathbb{R}^n)$ for every multi-index $\alpha \in \mathbb{N}^n$ with $0 \leq |\alpha| \leq n-1$.

(iii) For every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \tilde{v}(x) (-\Delta)^{\frac{n}{2}} \varphi(x) dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} \tilde{v}(x) (-\Delta)^{\frac{1}{2}} \varphi(x) dx = \int_{\mathbb{R}^n} \varphi(x) f(x) dx,$$

that is \tilde{v} solves (5) in the sense of Definition 1.1 and 1.2.

Proof. Proof of (i) is trivial.

To prove (ii) first we consider $0 < |\alpha| \leq n-1$ and we estimate

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{|D^\alpha \tilde{v}(x)|}{1+|x|^{n+1}} dx \\ &\leq C \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|^{n+1})|x-y|^{|\alpha|}} dx \right) dy \\ &= C \int_{\mathbb{R}^n} |f(y)| \left(\int_{B_1(y)} \frac{dx}{(1+|x|^{n+1})|x-y|^{|\alpha|}} + \int_{\mathbb{R}^n \setminus B_1(y)} \frac{dx}{(1+|x|^{n+1})|x-y|^{|\alpha|}} \right) dy \\ &\leq C \int_{\mathbb{R}^n} |f(y)| \left(\int_{B_1(y)} \frac{dx}{|x-y|^{|\alpha|}} + \int_{\mathbb{R}^n \setminus B_1(y)} \frac{dx}{(1+|x|^{n+1})} \right) dy \\ &< \infty. \end{aligned}$$

The case when $\alpha = 0$ follows from

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|\tilde{v}(x)|}{1+|x|^{n+1}} dx &\leq \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \frac{1}{1+|x|^{n+1}} \left(\int_{\mathbb{R}^n} \left| \log \frac{1+|y|}{|x-y|} \right| |f(y)| dy \right) dx \\
&= \frac{1}{\gamma_n} \int_{\mathbb{R}^n} |f(y)| \left(\int_{|x-y|>1} \frac{1}{1+|x|^{n+1}} \left| \log \frac{1+|y|}{|x-y|} \right| dx + \int_{|x-y|<1} \frac{1}{1+|x|^{n+1}} \left| \log \frac{1+|y|}{|x-y|} \right| dx \right) dy \\
&\leq \frac{1}{\gamma_n} \int_{\mathbb{R}^n} |f(y)| \left(\int_{|x-y|>1} \frac{\log(2+|x|)}{1+|x|^{n+1}} dx + \int_{|x-y|<1} \left(\frac{\log(2+|x|)}{1+|x|^{n+1}} + |\log|x-y|| \right) dx \right) dy \\
&= \frac{1}{\gamma_n} \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \frac{\log(2+|x|)}{1+|x|^{n+1}} dx + \|\log(\cdot)\|_{L^1(B_1)} \right) dy \\
&< \infty,
\end{aligned}$$

where in the first inequality we used

$$\frac{1}{1+|x|} \leq \frac{1+|y|}{|x-y|} \leq 2+|x|, \quad 1+|y| \leq 2+|x| \quad \text{for } |x-y| \geq 1.$$

(iii) follows from integration by parts and Lemma A.2. \square

Lemma 2.4 *Let u be a solution of (5) with $f \in L^1(\mathbb{R}^n)$ in the sense of Definition 1.2. Let \tilde{v} be given by (12). Then $p := u - \tilde{v}$ is a polynomial of degree at most $n-1$.*

Proof. Let us consider a function $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. We set

$$\varphi := \mathcal{F}^{-1} \left(\frac{\bar{\psi}}{|\xi|^n} \right) \in \mathcal{S}(\mathbb{R}^n), \quad \bar{\psi}(x) := \psi(-x), \quad x \in \mathbb{R}^n.$$

Now the growth assumption to u in Definition 1.2 implies that u is a tempered distribution and at the same time the function v is also a tempered distribution thanks to Lemma 2.3. Therefore $p \in L_{\frac{n}{2}}(\mathbb{R}^n)$ and $\hat{p} \in \mathcal{S}'(\mathbb{R}^n)$. Indeed,

$$\langle \hat{p}, \psi \rangle = \int_{\mathbb{R}^n} p \hat{\psi} dx = \int_{\mathbb{R}^n} p(x) (-\Delta)^{\frac{n}{2}} \varphi(x) dx = 0,$$

where the last equality follows from the Definition 1.2 and Lemma 2.3.

Thus \hat{p} is a tempered distribution with support $\hat{p} \subseteq \{0\}$ which implies that p is a polynomial and combining with $p \in L_{\frac{n}{2}}(\mathbb{R}^n)$ we conclude that degree of p is at most $n-1$. \square

Lemma 2.5 *Let u be a solution of (5) with $f \in L^1(\mathbb{R}^n)$ in the sense of Definition 1.1 and let \tilde{v} be given by (12). If u also satisfies*

$$\int_{B_R} u^+ dx = o(R^{2n}) \quad \text{or} \quad \int_{B_R} u^- dx = o(R^{2n}) \quad \text{as } R \rightarrow \infty, \quad (13)$$

then $p := u - \tilde{v}$ is a polynomial of degree at most $n-1$.

Proof. We have $\Delta^{\frac{n-1}{2}}p \in L_{\frac{1}{2}}(\mathbb{R}^n)$ and it satisfies

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}}p(-\Delta)^{\frac{1}{2}}\varphi dx = 0, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n), \quad (14)$$

thanks to Lemma 2.3. Moreover, by Schauder's estimate (see e.g. [14, Proposition 22]) for some $\alpha > 0$

$$\|(-\Delta)^{\frac{n-1}{2}}p\|_{C^{0,\alpha}(B_1)} \leq C\|(-\Delta)^{\frac{n-1}{2}}p\|_{L_{\frac{1}{2}}(\mathbb{R}^n)}.$$

Adapting the arguments in [14, Lemma 15] one can get that $(-\Delta)^{\frac{n-1}{2}}p$ is constant in \mathbb{R}^n and hence $(-\Delta)^{\frac{n+1}{2}}p = 0$ in \mathbb{R}^n . Noticing that $v \in L_{\frac{n}{2}}(\mathbb{R}^n)$ we conclude the proof by Lemma A.6 below. \square

Proposition 2.6 *Let $f \in L^1(\mathbb{R}^n)$. Then the following are equivalent:*

- (i) *u is a solution of (5) in the sense of Definition 1.2.*
- (ii) *u is a solution of (5) in the sense of Definition 1.1 and u satisfies (13).*

In particular, Definition 1.1 and Definition 1.2 are equivalent for the solutions of (1)-(2).

Proof. If p is a polynomial of degree at most $n-1$ then $p \in L_{\frac{n}{2}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} p(-\Delta)^{\frac{n}{2}}\varphi dx = \int_{\mathbb{R}^n} p(-\Delta)^{\frac{n-1}{2}}(-\Delta)^{\frac{1}{2}}\varphi dx = C_p \int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}}\varphi dx = 0, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where $C_p := (-\Delta)^{\frac{n-1}{2}}p$ is a constant and the second equality follows from integration by parts (which can be justified thanks to Lemma 2.2). Now the equivalence of (i) and (ii) follows immediately from Lemmas 2.3, 2.4 and 2.5. To conclude the lemma notice that the condition (2) implies

$$\int_{B_R} u^+ dx = \frac{1}{n} \int_{B_R} nu^+ dx \leq \frac{1}{n} \int_{B_R} e^{nu} dx \leq \frac{V}{n}.$$

\square

2.1 Proof of Theorem 1.1

First we write $(n-1)!e^{nu} = f_1 + f_2$ where $f_1 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $f_2 \in L^1(\mathbb{R}^n)$. Let us define the functions

$$u_i(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) f_i(y) dy, \quad x \in \mathbb{R}^n, i = 1, 2.$$

Then we have that $u_1 \in C^{n-1}(\mathbb{R}^n)$ and $u_2 \in W_{loc}^{n-1,1}(\mathbb{R}^n)$. Indeed, for $p \in \left(0, \frac{\gamma_n}{\|f_2\|}\right)$ using Jensen's inequality

$$\begin{aligned} \int_{B_R} e^{np|u_2|} dx &= \int_{B_R} \exp \left(\int_{\mathbb{R}^n} \frac{np\|f_2\|}{\gamma_n} \log \left(\frac{1+|y|}{|x-y|} \right) \frac{f_2(y)}{\|f_2\|} dy \right) dx \\ &\leq \int_{B_R} \int_{\mathbb{R}^n} \exp \left(\frac{np\|f_2\|}{\gamma_n} \log \left(\frac{1+|y|}{|x-y|} \right) \right) \frac{|f_2(y)|}{\|f_2\|} dy dx \\ &= \frac{1}{\|f_2\|} \int_{\mathbb{R}^n} |f_2(y)| \int_{B_R} \left(\frac{1+|y|}{|x-y|} \right)^{\frac{np\|f_2\|}{\gamma_n}} dx dy \\ &\leq C(n, p, \|f_2\|, R), \end{aligned} \tag{15}$$

where $\|\cdot\|$ denotes the $L^1(\mathbb{R}^n)$ norm. Moreover, by Lemma 2.3 (with $\tilde{v} = u_i$ and $f = f_i$) we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} u_i (-\Delta)^{\frac{1}{2}} \varphi dx = \int_{\mathbb{R}^n} f_i \varphi dx, \quad \text{for every } \varphi \in \mathcal{S}.$$

We set

$$u_3 := u - u_1 - u_2.$$

We claim that the function u_3 is smooth in \mathbb{R}^n whenever u is a solution of (1)-(2) in the sense of Definition 1.1 or 1.2. Then taking (15) into account we have $e^{nu} \in L_{loc}^p(\mathbb{R}^n)$ for every $p < \infty$ and hence $f_2 \in L_{loc}^p(\mathbb{R}^n)$. Therefore, for every $x \in B_R$ by Hölder's inequality

$$\begin{aligned} |u_2(x)| &\leq C \int_{|y| < 2R} \left| \log \left(\frac{1+|y|}{|x-y|} \right) \right| |f_2(y)| dy + C \int_{|y| \geq 2R} \left| \log \left(\frac{1+|y|}{|x-y|} \right) \right| |f_2(y)| dy \\ &\leq C (\log(1+2R) \|f_2\|_{L^1(B_{2R})} + \|\log(\cdot)\|_{L^2(B_{3R})} \|f_2\|_{L^2(B_{2R})}) + C \log(3R) \|f_2\|_{L^1(B_{2R}^c)}, \end{aligned}$$

and for every $0 < |\alpha| \leq n-1$ again by Hölder's inequality

$$\begin{aligned} |D^\alpha u_2(x)| &\leq C \int_{|y| < 2R} \frac{1}{|x-y|^{|\alpha|}} |f_2(y)| dy + C \int_{|y| \geq 2R} \frac{1}{|x-y|^{|\alpha|}} |f_2(y)| dy \\ &\leq C \| |\cdot|^{-|\alpha|} \|_{L^p(B_{3R})} \|f_2\|_{L^{p'}(B_{2R})} + C R^{-|\alpha|} \|f_2\|_{L^1(B_{2R}^c)}, \end{aligned}$$

where $p \in (1, \frac{n}{n-1})$. Thus $u_2 \in W_{loc}^{n-1,\infty}(\mathbb{R}^n)$ and by Sobolev embeddings we have $u_2 \in C^{n-2}(\mathbb{R}^n)$, which implies that $u = u_1 + u_2 + u_3 \in C^{n-2}(\mathbb{R}^n)$. Now to prove $u \in C^\infty(\mathbb{R}^n)$ we proceed by induction.

Set $\tilde{u} = u_1 + u_2$. Then for $0 < |\alpha| \leq n-1$

$$D^\alpha \tilde{u}(x) = \frac{(n-1)!}{\gamma_n} \int_{\mathbb{R}^n} D_x^\alpha \log \left(\frac{1+|y|}{|x-y|} \right) e^{nu(y)} dy =: \int_{\mathbb{R}^n} K_\alpha(x-y) e^{nu(y)} dy, \quad x \in \mathbb{R}^n.$$

Notice that the function K_α is smooth in $\mathbb{R}^n \setminus \{0\}$ and it also satisfies the estimate

$$|D^\beta K_\alpha(x)| \leq \frac{C_\alpha}{|x|^{|\alpha|+|\beta|}}, \quad \beta \in \mathbb{N}^n, x \in \mathbb{R}^n \setminus \{0\}.$$

We rewrite the function $D^\alpha \tilde{u}(x)$ as

$$\begin{aligned} D^\alpha \tilde{u}(x) &= \int_{\mathbb{R}^n} \eta(x-y) K_\alpha(x-y) e^{ny(y)} dy + \int_{\mathbb{R}^n} (1-\eta(x-y)) K_\alpha(x-y) e^{nu(y)} dy \\ &= \int_{\mathbb{R}^n} \eta(x-y) K_\alpha(x-y) e^{ny(y)} dy + \int_{\mathbb{R}^n} (1-\eta(y)) K_\alpha(y) e^{nu(x-y)} dy, \end{aligned}$$

where $\eta \in C^\infty(\mathbb{R}^n)$ satisfies

$$\eta(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ 1 & \text{if } |x| \geq 2. \end{cases}$$

If we assume $u \in C^k(\mathbb{R}^n)$ for some integer $k \geq 1$ then observing that $\eta K_\alpha \in C^\infty(\mathbb{R}^n)$, $D^\beta(\eta K_\alpha) \in L^\infty(\mathbb{R}^n)$ and $1-\eta$ is compactly supported, one has

$$D^{\alpha+\beta} \tilde{u}(x) = \int_{\mathbb{R}^n} D_x^\beta(\eta(x-y) K_\alpha(x-y)) e^{ny(y)} dy + \int_{\mathbb{R}^n} (1-\eta(y)) K_\alpha(y) D_x^\beta e^{nu(x-y)} dy, \quad |\beta| \leq k.$$

Thus $u \in C^{k+n-1}(\mathbb{R}^n)$ thanks to the claim that $u_3 \in C^\infty(\mathbb{R}^n)$, which proves our induction argument.

It remains to show that $u_3 \in C^\infty(\mathbb{R}^n)$ whenever u is a solution of (1)-(2) in the sense of Definition 1.1 or 1.2.

In the case of Definition 1.2 from Lemma 2.4 we have that u_3 is a polynomial of degree at most $n-1$ and hence it is smooth. On the other hand, if we consider Definition 1.1 then by Lemma 2.3 we get $\Delta^{\frac{n-1}{2}} u_3 \in L_{\frac{1}{2}}(\mathbb{R}^n)$ and it also satisfies (14) with $p = u_3$. Therefore, by [20, Proposition 2.22] we have $\Delta^{\frac{n-1}{2}} u_3 \in C^\infty(\mathbb{R}^n)$ which implies that $u_3 \in C^\infty(\mathbb{R}^n)$. \square

3 Classification of solutions

3.1 A fractional version of a lemma of Brézis and Merle

Theorem 3.2 below is a fractional version of a lemma of Brézis and Merle [2, Theorem 1], compare also [7, Theorem 5.1], which we shall later need in the proof of Lemma 3.8. Although, in our case Theorem 3.2 will be used in a smooth setting, here we shall prove it with more generality because of its independent interest. Before stating the theorem we need the following definition, partially inspired by [1, Section 3.3].

Definition 3.1 *Let Ω be a smooth bounded domain in \mathbb{R}^n . Assume $f \in L^1(\Omega)$ and $g_j \in L^1(\partial\Omega)$ for $j = 0, 1, \dots, \frac{n-3}{2}$. We say that $w \in L_{\frac{1}{2}}(\mathbb{R}^n)$ is a solution of*

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}} (-\Delta)^{\frac{1}{2}} w = f & \text{in } \Omega \\ (-\Delta)^j (-\Delta)^{\frac{1}{2}} w = g_j & \text{on } \partial\Omega, j = 0, 1, \dots, \frac{n-3}{2} \end{cases} \quad (16)$$

if w satisfies

$$\int_{d(x, \partial\Omega) < 2, x \in \Omega^c} \frac{|w(x)|}{\sqrt{\delta(x)}} dx < \infty, \quad (17)$$

and there exists a function $W \in L^1(\Omega)$ such that $(-\Delta)^{\frac{1}{2}}w = W$ in Ω , i.e.

$$\int_{\mathbb{R}^n} w(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{\Omega} W\varphi dx \quad \text{for every } \varphi \in T_1, \quad (18)$$

and the function W satisfies

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}}W = f & \text{in } \Omega \\ (-\Delta)^jW = g_j & \text{on } \partial\Omega, j = 0, 1, \dots, \frac{n-3}{2}, \end{cases} \quad (19)$$

i.e.

$$\int_{\Omega} W(-\Delta)^{\frac{n-1}{2}}\varphi dx = \int_{\Omega} f\varphi dx - \sum_{j=0}^{\frac{n-3}{2}} \int_{\partial\Omega} g_j \frac{\partial}{\partial\nu} (-\Delta)^{\frac{n-3}{2}-j}\varphi d\sigma \quad \text{for every } \varphi \in T_2,$$

where the spaces of test functions T_1 and T_2 are defined by

$$T_1 := \left\{ \varphi \in C^\infty(\Omega) \cap C^{\frac{1}{2}}(\mathbb{R}^n) : \begin{cases} (-\Delta)^{\frac{1}{2}}\varphi = \psi & \text{in } \Omega \\ \varphi = 0 & \text{on } \Omega^c \end{cases} \text{ for some } \psi \in C_c^\infty(\Omega), \right\},$$

and

$$T_2 := \left\{ \varphi \in C^{n-1}(\overline{\Omega}) : \Delta^j\varphi = 0 \text{ on } \partial\Omega, j = 0, 1, \dots, \frac{n-3}{2} \right\}.$$

Notice that the left hand side of (18) is well-defined thanks to the assumption (17) and Lemma 3.4 below.

Lemma 3.1 (Maximum Principle) *Let w be a solution of (16) with $f, g_j \geq 0$ in the sense of Definition 3.1. If $w \geq 0$ on Ω^c then $w \geq 0$ in Ω .*

Proof. First notice that the conditions $f \geq 0, g_j \geq 0$ implies that $W \geq 0$ in Ω , where $W \in L^1(\Omega)$ is a solution of (19). Now consider a test function $\psi \in C_c^\infty(\Omega)$ such that $\psi \geq 0$ in Ω . Let $\varphi \in T_1$ be the solution of $(-\Delta)^{\frac{1}{2}}\varphi = \psi$ in Ω . Then by classical maximum principle one has $\varphi \geq 0$ in Ω . Since the constant $C_{n, \frac{1}{2}} > 0$ in Proposition A.1 we get

$$(-\Delta)^{\frac{1}{2}}\varphi(x) < 0 \quad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega},$$

and from (18)

$$\int_{\Omega} w\psi dx = \int_{\Omega} w(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{\Omega} W\varphi dx - \int_{\Omega^c} w(-\Delta)^{\frac{1}{2}}\varphi dx \geq 0,$$

which completes the proof. \square

Theorem 3.2 *Let $f \in L^1(B_R)$. Let $u \in L^1(B_R)$ be a solution of (16) (in the sense of Definition 3.1) with $g_j = 0$ for $j = 0, 1, \dots, \frac{n-3}{2}$ and $u = 0$ on B_R^c . Then for any $p \in \left(0, \frac{\gamma_n}{\|f\|_{L^1(B_R)}}\right)$*

$$\int_{B_R} e^{np|u|} dx \leq C(p, R).$$

Proof. We set

$$\overline{W}(x) = \int_{B_R} \Psi(x-y)|f(y)|dy \quad x \in \mathbb{R}^n,$$

where

$$\Psi(x) := \frac{\Gamma(\frac{1}{2})}{n2^{n-2}|B_1|\Gamma(\frac{n}{2})\left(\frac{n-3}{2}\right)!|x|},$$

is a fundamental solution of $(-\Delta)^{\frac{n-1}{2}}$ in \mathbb{R}^n (see [9, Section 2.6]). Then $\overline{W} \in L^1(B_R)$ satisfies

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}}\overline{W} = |f| & \text{in } B_R \\ (-\Delta)^j\overline{W} \geq 0 & \text{on } \partial B_R, j = 0, 1, \dots, \frac{n-3}{2}, \end{cases}$$

and by maximum principle $\overline{W} \geq |W|$ in B_R , where $W \in L^1(B_R)$ is a solution of (19). Let us define

$$\overline{u}(x) := \Phi * (\overline{W}\chi_{B_R})(x) = \frac{(\frac{n-3}{2})!}{2\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} \overline{W}(y)\chi_{B_R}(y)dy, \quad x \in \mathbb{R}^n,$$

where Φ is given in Lemma A.2 below. Noticing

$$\frac{1}{\gamma_n} = |S^{n-1}| \frac{\Gamma(\frac{1}{2})}{n2^{n-2}|B_1|\Gamma(\frac{n}{2})\left(\frac{n-3}{2}\right)!} \frac{(\frac{n-3}{2})!}{2\pi^{\frac{n+1}{2}}},$$

in view of Lemma 3.3 below one has

$$|\overline{u}(x)| \leq C + \frac{1}{\gamma_n} \int_{|y|<R} |f(y)||\log|x-y||dy, \quad x \in \mathbb{R}^n,$$

which yields

$$\overline{u} \in L_{loc}^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \setminus B_{R+\delta}), \quad q \in [1, \infty), \delta > 0.$$

Moreover, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{B_R} \overline{W}\varphi dx = \int_{\mathbb{R}^n} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{B_R} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx + \int_{B_R^c} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx, \quad (20)$$

thanks to Lemma A.2 below.

We claim that (20) holds for $\varphi \in T_1$. Then for any $\varphi \in T_1$ with $\varphi \geq 0$

$$\int_{B_R} (\overline{u} \pm u)(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{B_R} \underbrace{(\overline{W} \pm W)}_{\geq 0} \varphi dx - \int_{B_R^c} \underbrace{\overline{u}(-\Delta)^{\frac{1}{2}}\varphi}_{\leq 0} dx \geq 0,$$

and by maximum principle one has $\overline{u} \geq |u|$ in B_R and the lemma follows at once.

To prove the claim we consider a mollifying sequence $\varphi_k := \varphi * \rho_k$, where $\rho_k(x) = k^n \rho(kx)$. Then (see [1, Section A])

$$(-\Delta)^{\frac{1}{2}}\varphi_k(x) = \varphi * (-\Delta)^{\frac{1}{2}}\rho_k(x) \quad x \in \mathbb{R}^n,$$

and

$$(-\Delta)^{\frac{1}{2}}\varphi_k(x) = \rho_k * (-\Delta)^{\frac{1}{2}}\varphi(x), \quad \text{dist}(x, \partial B_R) > \frac{1}{k}. \quad (21)$$

Then the uniform convergence of φ_k to φ imply

$$\int_{B_R} \overline{W}\varphi_k dx \xrightarrow{k \rightarrow \infty} \int_{B_R} \overline{W}\varphi dx.$$

Using the uniform convergence of $(-\Delta)^{\frac{1}{2}}\varphi_k$ to $(-\Delta)^{\frac{1}{2}}\varphi$ on the compact sets in B_R and the fact that $\text{supp}((-\Delta)^{\frac{1}{2}}\varphi|_{B_R}) \subseteq B_R$ we get

$$\int_{B_R} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi_k dx \xrightarrow{k \rightarrow \infty} \int_{B_R} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx.$$

It remains to verify that

$$\int_{B_R^c} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi_k dx \xrightarrow{k \rightarrow \infty} \int_{B_R^c} \overline{u}(-\Delta)^{\frac{1}{2}}\varphi dx,$$

which follows immediately from

$$(-\Delta)^{\frac{1}{2}}\varphi_k \xrightarrow{k \rightarrow \infty} (-\Delta)^{\frac{1}{2}}\varphi \text{ in } L^q(B_{R+1} \setminus B_R), \text{ for some } q > 1, \quad (22)$$

and

$$(-\Delta)^{\frac{1}{2}}\varphi_k \xrightarrow{k \rightarrow \infty} (-\Delta)^{\frac{1}{2}}\varphi \text{ in } L^1(B_{R+1}^c). \quad (23)$$

With the help of Lemma 3.4 below and (21) one can get (23). To conclude (22) first notice that $(-\Delta)^{\frac{1}{2}}\varphi_k$ converges to $(-\Delta)^{\frac{1}{2}}\varphi$ point-wise and that $(-\Delta)^{\frac{1}{2}}\varphi \in L^q(B_{R+1} \setminus B_R)$ for any $q \in [1, 2)$ thanks to Lemma 3.4 below. By [15, Theorem 1.9 (Missing term in Fatou's lemma)] it is sufficient to show that for some $q > 1$

$$\int_{R < |x| < R+1} |(-\Delta)^{\frac{1}{2}}\varphi_k(x)|^q dx \leq \int_{R < |x| < R+1} |(-\Delta)^{\frac{1}{2}}\varphi(x)|^q dx + o(1),$$

where $o(1) \rightarrow 0$ as $k \rightarrow \infty$. Now using the estimate (see for instance [1, Section A])

$$|(-\Delta)^{\frac{1}{2}}\rho_k(x)| \leq Ck^{n+1} \quad x \in \mathbb{R}^n,$$

and fixing t and q such that

$$\frac{2n}{2n+1} < t < 1, \quad 1 < q < \min \left\{ \frac{1+nt}{t+nt}, \frac{2nt+t+2}{2n+2} \right\},$$

we bound

$$\begin{aligned}
& \int_{R < |x| < R+1} |(-\Delta)^{\frac{1}{2}} \varphi_k(x)|^q dx = \int_{R < |x| < R+\frac{1}{k}} |(-\Delta)^{\frac{1}{2}} \varphi_k(x)|^q dx + \int_{R+\frac{1}{k} < |x| < R+1} |(-\Delta)^{\frac{1}{2}} \varphi_k(x)|^q dx \\
& = \int_{R < |x| < R+\frac{1}{k}} |\varphi * (-\Delta)^{\frac{1}{2}} \rho_k(x)|^q dx + \int_{R+\frac{1}{k} < |x| < R+1} |\rho_k * (-\Delta)^{\frac{1}{2}} \varphi(x)|^q dx \\
& \leq \|\varphi\|_{L^1}^{q-1} \int_{R < |x| < R+\frac{1}{k}} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} \rho_k(y)|^q |\varphi(x-y)| dy dx \\
& \quad + \int_{R+\frac{1}{k} < |x| < R+1} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{1}{2}} \varphi(y)|^q \rho_k(x-y) dy dx \\
& = \int_{R < |y| < R+1+\frac{1}{k}} |(-\Delta)^{\frac{1}{2}} \varphi(y)|^q + \|\varphi\|_{L^1}^{q-1} \int_{R < |x| < R+\frac{1}{k}} \int_{|y| > \frac{1}{k^t}} |(-\Delta)^{\frac{1}{2}} \rho_k(y)|^q |\varphi(x-y)| dy dx \\
& \quad + \|\varphi\|_{L^1}^{q-1} \int_{R < |x| < R+\frac{1}{k}} \int_{|x-y| < R, |y| < \frac{1}{k^t}} |(-\Delta)^{\frac{1}{2}} \rho_k(y)|^q |\varphi(x-y)| dy dx \\
& \leq \int_{R < |y| < R+1+\frac{1}{k}} |(-\Delta)^{\frac{1}{2}} \varphi(y)|^q + C \|\varphi\|_{L^1}^{q-1} k^{t(q+nq-n)-1} + C \|\varphi\|_{L^1}^{q-1} k^{q(n+1)-nt-\frac{t}{2}-1} \\
& = \int_{R < |y| < R+1} |(-\Delta)^{\frac{1}{2}} \varphi(y)|^q + o(1),
\end{aligned}$$

where in the last inequality we have used (for the second term)

$$\begin{aligned}
\int_{|x| > \frac{1}{k^t}} |(-\Delta)^{\frac{1}{2}} \rho_k(x)|^q dx &= \int_{|x| > \frac{1}{k^t}} \left| C_{1/2} P.V. \int_{\mathbb{R}^n} \frac{\rho_k(x) - \rho_k(y)}{|x-y|^{n+1}} dy \right|^q dx \\
&\leq C \int_{|x| > \frac{1}{k^t}} \int_{|y| < 1} \frac{\rho(y)^q}{|x - \frac{y}{k}|^{nq+q}} dy dx \\
&\leq C \int_{|y| < 1} \int_{|x| > \frac{1}{k^t}} \frac{1}{|x|^{nq+q}} dx dy \\
&\leq C k^{t(q+nq-n)}.
\end{aligned}$$

□

Lemma 3.3 *Let Ω be a domain in \mathbb{R}^n . Let p and q be two positive real numbers. Then*

$$\int_{\Omega} \frac{dy}{|x-y|^{n+p}} \leq \frac{|S^{n-1}|}{p} \frac{1}{\delta(x)^p}, \quad \text{if } \delta(x) := \text{dist}(x, \Omega) > 0,$$

and

$$\int_{\Omega} \frac{dz}{|x-z|^p |y-z|^q} \leq \frac{C_{n,p,q}}{|x-y|^{p+q-n}}, \quad \text{if } p+q > n, p < n, q < n, x \neq y,$$

where the constant $C_{n,p,q}$ is given by (an explicit formula can be found in [15, Section 5.10])

$$C_{n,p,q} = \int_{\mathbb{R}^n} \frac{dz}{|z|^p |e_1 - z|^q}.$$

In addition if we also assume that the domain Ω is bounded then

$$\int_{\Omega} \frac{dy}{|x-y|^n} \leq |\Omega| + |S^{n-1}| |\log \delta(x)| \quad \text{if } \delta(x) > 0,$$

and

$$\int_{\Omega} \frac{dz}{|x-z|^p |y-z|^q} \leq C + |S^{n-1}| |\log(|x-y|)|, \quad \text{if } p+q=n, p < n, q < n, x \neq y.$$

Proof. Let us denote the set $\{y-x : y \in \Omega\}$ by $\Omega-x$. Using a change of variable $z \mapsto z-x$ and setting $w = y-x$ we have

$$\int_{\Omega} \frac{dz}{|x-z|^p |y-z|^q} = \int_{\Omega-x} \frac{dz}{|z|^p |w-z|^q} =: I.$$

If $p+q > n$ then changing the variable $z \mapsto |w|z$ one has

$$I = \frac{1}{|w|^{p+q-n}} \int_{\frac{1}{|w|}(\Omega-x)} \frac{dz}{|z|^p |\frac{w}{|w|}-z|^q} \leq \frac{1}{|w|^{p+q-n}} \int_{\mathbb{R}^n} \frac{dz}{|z|^p |\frac{w}{|w|}-z|^q} = \frac{C_{n,p,q}}{|w|^{p+q-n}}.$$

In the case when $p+q = n$, we split the domain $\Omega-x$ into two disjoint domains:

$$\Omega_1 := (\Omega-x) \cap B_1, \quad \Omega_2 = (\Omega-x) \cap B_1^c.$$

Then

$$I = \sum_{i=1}^2 I_i, \quad I_i := \int_{\Omega_i} \frac{dz}{|z|^p |w-z|^q}.$$

Since Ω_2 is bounded and $q < n$, we have

$$I_2 \leq \int_{\Omega_2} \frac{dz}{|w-z|^q} \leq C.$$

Now using

$$\frac{1}{|\frac{w}{|w|}-z|} \leq \frac{1}{|z|} \left(1 + \frac{2}{|z|}\right) \quad \text{for } |z| \geq 2,$$

and

$$(1+x)^q \leq 1 + C_q x \quad \text{for } x \in (0,1),$$

we bound

$$\begin{aligned} I_1 &\leq \int_{B_1} \frac{dz}{|z|^p |w-z|^q} = \int_{|z| \leq \frac{1}{|w|}} \frac{dz}{|z|^p |\frac{w}{|w|}-z|^q} \\ &\leq \underbrace{\int_{|z| \leq 2} \frac{dz}{|z|^p |\frac{w}{|w|}-z|^q}}_{\leq C} + \int_{2 < |z| \leq \frac{1}{|w|}} \frac{1}{|z|^n} \left(1 + \frac{2}{|z|}\right)^q dz \\ &\leq \int_{2 < |z| \leq \frac{1}{|w|}} \frac{1}{|z|^n} \left(1 + \frac{C}{|z|}\right) dz \\ &\leq C + |S^{n-1}| |\log |w||. \end{aligned}$$

Finally, we conclude the lemma by showing that for $x \in \mathbb{R}^n \setminus \overline{\Omega}$

$$\int_{\Omega} \frac{dy}{|x-y|^{n+p}} \leq \int_{|z|>\delta(x)} \frac{dy}{|z|^{n+p}} = \frac{|S^{n-1}|}{p} \frac{1}{\delta(x)^p}, \quad p > 0,$$

and

$$\int_{\Omega} \frac{dy}{|x-y|^n} \leq |\Omega| + \int_{\Omega \cap B_1(x)} \frac{dy}{|x-y|^n} \leq |\Omega| + \int_{\delta(x) < |z| < 1} \frac{dy}{|z|^n} = |\Omega| + |S^{n-1}| |\log \delta(x)|.$$

□

Lemma 3.4 *Let Ω be a bounded domain in \mathbb{R}^n . Let $\varphi \in C^{k,\sigma}(\mathbb{R}^n)$ for some nonnegative integer k and $0 \leq \sigma \leq 1$ be such that $\varphi = 0$ on $\mathbb{R}^n \setminus \Omega$. Then for $0 < s < 1$ and for $x \in \mathbb{R}^n \setminus \overline{\Omega}$*

$$|(-\Delta)^s \varphi(x)| \leq C \begin{cases} \min\{\max\{1, \delta(x)^{-2s+k+\sigma}\}, \delta(x)^{-n-2s}\} & \text{if } k + \sigma \neq 2s \\ \min\{|\log \delta(x)|, \delta(x)^{-n-2s}\} & \text{if } k + \sigma = 2s, \end{cases}$$

where $\delta(x) := \text{dist}(x, \Omega)$.

Proof. We claim that

$$|\varphi(y)| \leq C|x-y|^{k+\sigma}, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}, y \in \Omega,$$

which can be verified using the Taylor's expansion

$$\varphi(y) = \sum_{|\alpha| \leq k-1} \frac{1}{\alpha!} \underbrace{D^\alpha \varphi(x)}_{=0} (y-x)^\alpha + \sum_{|\beta|=k} \frac{|\beta|}{\beta!} (y-x)^\beta \int_0^1 (1-t)^{|\beta|-1} D^\beta \varphi(x+t(y-x)) dt,$$

and

$$|D^\beta \varphi(x+t(y-x))| = |D^\beta \varphi(x+t(y-x)) - D^\beta \varphi(x)| \leq C|t(x-y)|^\sigma \leq C|x-y|^\sigma.$$

Therefore, by Proposition A.1

$$|(-\Delta)^s \varphi(x)| = \left| C_{n,s} \int_{\Omega} \frac{\varphi(y)}{|x-y|^{n+2s}} dy \right| \leq C \int_{\Omega} \frac{dy}{|x-y|^{n+2s-k-\sigma}}, \quad x \in \mathbb{R}^n \setminus \overline{\Omega},$$

and

$$|(-\Delta)^s \varphi(x)| \leq C \int_{\Omega} \frac{|\varphi(y)|}{|x-y|^{n+2s}} dy \leq C \int_{\Omega} \frac{|\varphi(y)|}{\delta(x)^{n+2s}} dy \leq \frac{C}{\delta(x)^{n+2s}}, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Now the proof follows at once from Lemma 3.3.

□

3.2 Proof of Theorem 1.2

First we study the asymptotic behavior of v defined in (9).

Lemma 3.5 *Let u be a smooth solution of (1)-(2) and let v be given by (9). Then there exists a constant $C > 0$ such that*

$$v(x) \geq -\alpha \log |x| - C, \quad |x| \geq 4.$$

Proof. The proof follows as in the proof of [16, Lemma 2.1]. \square

A consequence of the above lemma is the following Proposition, compare Lemmas 2.4, 2.5.

Proposition 3.6 *Let u be a smooth solution of (1)-(2) in the sense of Definition 1.1 or 1.2 and let v be defined by (9). Then the function*

$$P(x) := u(x) - v(x), \quad x \in \mathbb{R}^n,$$

is a polynomial of degree at most $n - 1$ and P is bounded above.

Proof. Since (2) implies (13), by Lemmas 2.4 and 2.5 we have that P is a polynomial of degree at most $n - 1$. On the other hand, using Lemma 3.5 one can get that P is bounded above (the proof is very similar to [17, Lemma 11]). \square

Lemma 3.7 *Let $n \geq 3$ be an odd integer and let u be a smooth solution of (1)-(2) and v be given by (9). Then*

(i) $v \in C^\infty(\mathbb{R}^n)$ and $D^\alpha v \in L_{\frac{1}{2}}(\mathbb{R}^n)$ for every multi-index $\alpha \in \mathbb{N}^n$ with $0 \leq |\alpha| \leq n - 1$.

(ii) *There exists a constants $C > 0$ such that*

$$\int_{\partial B_4(x)} |(-\Delta)^j (-\Delta)^{\frac{1}{2}} v(y)| d\sigma(y) \leq C \text{ for every } x \in \mathbb{R}^n, j = 0, 1, 2, \dots, \frac{n-3}{2}.$$

(iii) v is a pointwise solution of

$$(-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{n-1}{2}} v = (n-1)! e^{nu} \quad \text{in } \mathbb{R}^n.$$

(iv) v solves (16) with $f = (n-1)! e^{nu}$ and $g_j = (-\Delta)^j (-\Delta)^{\frac{1}{2}} v$ for every $j = 0, 1, 2, \dots, \frac{n-3}{2}$.

Proof. We divide the proof into several steps.

Step 1. From Proposition 3.6 we have the smoothness of v and by Lemma 2.3 we get $D^\alpha v \in L_{\frac{1}{2}}(\mathbb{R}^n)$ for every multi-index $\alpha \in \mathbb{N}^n$ with $0 \leq |\alpha| \leq n - 1$.

Step 2. In this step we use (i) to prove (ii). In fact by Lemmas A.3, A.5, below we have

$$\begin{aligned} \int_{\partial B_4(x)} |(-\Delta)^j (-\Delta)^{\frac{1}{2}} v(y)| d\sigma(y) &= \int_{\partial B_4(x)} |(-\Delta)^{\frac{1}{2}} (-\Delta)^j v(z)| d\sigma(z) \\ &\leq C \int_{\partial B_4(x)} \int_{\mathbb{R}^n} \frac{e^{nu(y)}}{|y-z|^{2j+1}} dy d\sigma(z) \\ &= C \int_{\mathbb{R}^n} e^{nu(y)} \int_{\partial B_4(x)} \frac{1}{|y-z|^{2j+1}} d\sigma(z) dy \\ &\leq C. \end{aligned}$$

Step 3. We claim that for $g \in C^\infty(\mathbb{R}^n) \cap L_{\frac{1}{2}}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} g \varphi dx = \int_{\mathbb{R}^n} g (-\Delta)^{\frac{1}{2}} \varphi dx \text{ for every } \varphi \in C_c^\infty(\mathbb{R}^n).$$

To prove the claim we consider a approximating sequence

$$g_k(x) := g(x) \psi\left(\frac{x}{k}\right), \quad \psi \in C^\infty(\mathbb{R}^n), \quad \psi(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2. \end{cases}$$

Then $g_k \in \mathcal{S}(\mathbb{R}^n)$ and hence

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} g_k \varphi dx = \int_{\mathbb{R}^n} g_k (-\Delta)^{\frac{1}{2}} \varphi dx.$$

Now the claim follows from the locally uniform convergence of $(-\Delta)^{\frac{1}{2}} g_k$ to $(-\Delta)^{\frac{1}{2}} g$ and the $L_{\frac{1}{2}}(\mathbb{R}^n)$ convergence of g_k to g .

Step 4. Using *Step 3* with $g = (-\Delta)^{\frac{n-1}{2}} v$ we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{n-1}{2}} v \varphi dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} v (-\Delta)^{\frac{1}{2}} \varphi dx = (n-1)! \int_{\mathbb{R}^n} e^{nu} \varphi dx,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, which implies (iii).

To complete (iv) it suffices to show that $W := (-\Delta)^{\frac{1}{2}} v \in C^\infty(\mathbb{R}^n)$ and it satisfies (17)-(19) with $w = v$.

The smoothness of v implies $W \in C^\infty(\mathbb{R}^n)$ and (17). Moreover, using integration by parts (see [1, Proposition 1.2.1]) one can get (18).

One must notice that the function u in [1, Proposition 1.2.1] is in $C^{1+\varepsilon}(\Omega) \cap L^\infty(\mathbb{R}^n)$ but still we can use it since our function $v \in C^\infty(\mathbb{R}^n) \cap L_{\frac{1}{2}}(\mathbb{R}^n)$.

Finally, we prove (19) by showing that W is a classical solution of (19). Since W is smooth in \mathbb{R}^n clearly it satisfies the boundary conditions. Using *step 3* (with $g = v$) and Lemma 2.3 (with $f = (n-1)!e^{nu}$) we have for every $\varphi \in C_c^\infty(\Omega)$

$$\begin{aligned} \int_{\Omega} (-\Delta)^{\frac{n-1}{2}} W \varphi dx &= \int_{\Omega} W (-\Delta)^{\frac{n-1}{2}} \varphi dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} v (-\Delta)^{\frac{n-1}{2}} \varphi dx \\ &= \int_{\mathbb{R}^n} v (-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{n-1}{2}} \varphi dx = (n-1)! \int_{\mathbb{R}^n} e^{nu} \varphi dx, \end{aligned}$$

that is

$$(-\Delta)^{\frac{n-1}{2}} W = (n-1)!e^{nu} \quad \text{in } \Omega.$$

□

The following lemma is the crucial part in the proof of Theorem 1.2.

Lemma 3.8 *Let u be a smooth solution of (1)-(2) and v be given by (9). Then for any $\varepsilon > 0$ there exists $R > 0$ such that for $|x| > R$*

$$v(x) \leq (-\alpha + \varepsilon) \log |x|.$$

Proof. Step 1. For any $\varepsilon > 0$ there exists a $R > 0$ such that for $|x| \geq R$

$$v(x) \leq (-\alpha + \frac{\varepsilon}{2}) \log |x| - \frac{(n-1)!}{2} \int_{B_1(x)} \log |x-y| e^{nu(y)} dy. \quad (24)$$

The proof of (24) is very similar to the proof of [16, Lemma 2.4]. As a consequence of (24) using Jensen's inequality we have the following estimate

$$\|v^+\|_{L^p(\mathbb{R}^n)} \leq |\alpha - \frac{\varepsilon}{2}| \|\log\|_{L^p(B_1)} + \frac{(n-1)!}{2} \|e^{nu}\|_{L^1(\mathbb{R}^n)} \|\log\|_{L^p(B_1)}, \quad 1 \leq p < \infty. \quad (25)$$

Step 2. We claim that there exists $p > 1$ and $C > 0$ independent of x_0 such that $\|e^{nu}\|_{L^p(B_1(x_0))} \leq C$. Then using Hölder inequality one can bound the second term on the right hand side of (24) uniformly in x and that completes the proof of the lemma.

To prove the claim, first notice that it is sufficient to consider $x_0 \in \mathbb{R}^n \setminus B_R$ for any fixed $R > 0$. We choose $R > 0$ large enough such that

$$(n-1)! \|e^{nu}\|_{L^1(B_{R-1}^c)} < \frac{\gamma_n}{2}.$$

Let $w \in C^0(\mathbb{R}^n)$ be the solution of

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}} (-\Delta)^{\frac{1}{2}} w = (n-1)! e^{nu} & \text{in } B_4(x_0) \subset \mathbb{R}^n \\ (-\Delta)^j (-\Delta)^{\frac{1}{2}} w = 0 & \text{on } \partial B_4(x_0), \text{ for } j = 0, 1, \dots, \frac{n-3}{2} \\ w = 0 & \text{on } \mathbb{R}^n \setminus B_4(x_0), \end{cases}$$

in the sense of Definition 3.1. Since u is smooth by Schauder's estimates and bootstrap argument we have $W = (-\Delta)^{\frac{1}{2}} w \in C^\infty(\overline{B_4(x_0)})$ which solves (16) with $f = (n-1)! e^{nu}$ and $g_j = (-\Delta)^j (-\Delta)^{\frac{1}{2}} v$ for every $j = 0, 1, 2, \dots, \frac{n-3}{2}$. Then using Green's representation formula (see [3, Theorem 3]) one can get $w \in C^0(\mathbb{R}^n)$ (in fact $w \in C^{\frac{1}{2}}(\mathbb{R}^n)$, see [19]), which is the pointwise continuous unique solution of

$$(-\Delta)^{\frac{1}{2}} w = W \quad \text{in } B_4(x_0), \quad w = 0 \quad \text{on } B_4(x_0)^c.$$

Moreover, w satisfies (18) thanks to [1, Proposition 3.3.3].

We set $h = v - w$. Then we have that $h \in C^0(\mathbb{R}^n)$, $(-\Delta)^{\frac{1}{2}} h \in C^\infty(\overline{B_4(x_0)})$ and

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}} (-\Delta)^{\frac{1}{2}} h = 0 & \text{in } B_4(x_0) \\ (-\Delta)^j (-\Delta)^{\frac{1}{2}} h = (-\Delta)^j (-\Delta)^{\frac{1}{2}} v & \text{on } \partial B_4(x_0), j = 0, 1, \dots, \frac{n-3}{2} \\ h = v & \text{on } \mathbb{R}^n \setminus B_4(x_0), \end{cases} \quad (26)$$

thanks to Lemma 3.7. Indeed, by Lemma 3.9 below there exists a constant $C > 0$ independent of the choice of $x_0 \in \mathbb{R}^n$ such that

$$h(x) \leq C \quad \text{for every } x \in B_1(x_0).$$

Hence by Proposition 3.6

$$u = v + P \leq C + h + w \leq C + w,$$

and by Theorem 3.2 we have the proof. □

A simple consequence of Lemma 3.8 is that

$$\lim_{|x| \rightarrow \infty} u(x) = -\infty, \quad (27)$$

thanks to Proposition 3.6. Using (27) one can show that

$$\lim_{|x| \rightarrow \infty} D^\beta v(x) = 0 \text{ for every } \beta \in \mathbb{N}^n \text{ with } 0 < |\beta| < n - 1.$$

Now the proof of Theorem 1.2 follows at once from Lemmas 3.5, 3.8 and Proposition 3.6.

Lemma 3.9 *Let $h \in C^0(\mathbb{R}^n)$ be given by (26). Then there exists a constant $C > 0$ (independent of x_0) such that*

$$h(x) \leq C, \text{ for every } x \in B_1(x_0).$$

Proof. Let us write $h = h_1 + h_2$ where $h_1, h_2 \in C^0(\mathbb{R}^n)$ be such that

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_1 = (-\Delta)^{\frac{1}{2}} h & \text{in } B_4(x_0) \\ h_1 = 0 & \text{on } B_4(x_0)^c, \end{cases}$$

and

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_2 = 0 & \text{in } B_4(x_0) \\ h_2 = h = v & \text{on } B_4(x_0)^c. \end{cases}$$

Let $h_3 \in C^0(\mathbb{R}^n)$ be such that

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_3 = 0 & \text{in } B_4(x_0) \\ h_3 = v^+ & \text{on } B_4(x_0)^c. \end{cases}$$

Then by maximum principle

$$h_2 \leq h_3 \text{ on } \mathbb{R}^n.$$

Without loss of generality we can assume that $x_0 = 0$. Then the Poisson formula gives (see [3, Theorem 1])

$$h_3(x) = \int_{|y| > 4} P(x, y) v^+(y) dy, \quad x \in B_4,$$

where

$$P(x, y) = C_n \left(\frac{16 - |x|^2}{|y|^2 - 16} \right)^{\frac{1}{2}} \frac{1}{|x - y|^n}.$$

Now for $x \in B_2$ by Hölder's inequality we get

$$\begin{aligned} |h_3(x)| &\leq C \int_{|y| > 4} \left(\frac{1}{|y|^2 - 16} \right)^{\frac{1}{2}} \frac{1}{|y|^n} v^+(y) dy \\ &\leq C \left(\int_{|y| > 4} v^+(y)^3 dy \right)^{\frac{1}{3}} \left(\int_{|y| > 4} \frac{1}{(|y|^2 - 16)^{\frac{3}{4}} |y|^{\frac{3n}{2}}} dy \right)^{\frac{2}{3}} \\ &\leq C \|v^+\|_{L^3(\mathbb{R}^n)} \leq C, \end{aligned}$$

where the last inequality follows from (25). By Lemma 3.10 below we have

$$h \leq C, \text{ for every } x \in B_1(x_0),$$

where C being independent of x_0 . □

Lemma 3.10 *Let $h \in C^0(\mathbb{R}^n)$ solves (26). Let $h_1 \in C^0(\mathbb{R}^n)$ be the solution of*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} h_1 = (-\Delta)^{\frac{1}{2}} h & \text{in } B_4(x_0) \\ h_1 = 0 & \text{on } B_4(x_0)^c. \end{cases}$$

Then there exists a constant $C = C(n)$ such that

$$\|h_1\|_{L^\infty(B_1(x_0))} \leq C.$$

Proof. We assume that $x_0 = 0$. Using Green's representation formula (see [3, Theorem 3]) the solution is given by

$$h_1(x) = \int_{B_4} G_2(x, y) (-\Delta)^{\frac{1}{2}} h(y) dy, \quad x \in B_4,$$

where

$$G_2(x, y) = C_n |x - y|^{1-n} \int_0^{r_0(x, y)} \frac{r^{\frac{1}{2}-1}}{(1+r)^{\frac{n}{2}}} dr, \quad r_0(x, y) = \frac{(16 - |x|^2)(16 - |y|^2)}{|x - y|^2}.$$

Since

$$\frac{r^{-\frac{1}{2}}}{(1+r)^{\frac{n}{2}}} \in L^1((0, \infty)),$$

we have

$$|G_2(x, y)| \leq C |x - y|^{1-n}.$$

For $|z| \leq 1$ using (26), Lemma 3.7 and Lemma A.4 below we bound

$$\begin{aligned} |h_1(z)| &\leq \int_{B_4} |G_2(z, y)| |(-\Delta)^{\frac{1}{2}} h(y)| dy \\ &\leq \sum_{i=0}^{\frac{n-3}{2}} \int_{B_4} |G_2(z, y)| \left(\int_{\partial B_4} \left| (-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x) \right| \left| \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(y, x) \right) \right| d\sigma(x) \right) dy \\ &\leq C \sum_{i=0}^{\frac{n-3}{2}} \int_{B_4} |z - y|^{1-n} \left(\int_{\partial B_4} \left| (-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x) \right| |x - y|^{1+2i-n} d\sigma(x) \right) dy \\ &= C \sum_{i=0}^{\frac{n-3}{2}} \int_{|x|=4} \left| (-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x) \right| \left(\int_{|y|<4} |z - y|^{1-n} |x - y|^{1+2i-n} dy \right) d\sigma(x) \\ &\leq C \sum_{i=0}^{\frac{n-3}{2}} \int_{|x|=4} \left| (-\Delta)^i (-\Delta)^{\frac{1}{2}} v(x) \right| d\sigma(x) \\ &\leq C. \end{aligned}$$

□

3.3 Proof of Theorem 1.3

One can verify easily that $(i) \Rightarrow (ii)-(vi)$. On the other hand, by Theorem 1.2 (ii) to (iv) are equivalent. Moreover, $(ii) \Rightarrow (i)$ thanks to [22, Theorem 4.1]. To show that $(v) \Rightarrow (i)$ and $(vi) \Rightarrow (i)$ one can follow the arguments in [17].

Finally to prove (11) we use [17, Theorem 6 and Lemma 3]. Since the polynomial P is bounded from above, $\deg(P)$ must be even and let it be $2k$. Then $\Delta^k P = C_0$ on \mathbb{R}^n and $\Delta^{k+1} P = 0$ on \mathbb{R}^n . By [17, Lemma 3] we have

$$\sum_{i=0}^k c_i R^{2i} \Delta^i P(0) = \frac{1}{|B_R|} \int_{B_R} P(x) dx \leq \sup_{\mathbb{R}^n} P \leq C, \text{ for every } R > 0,$$

where the constants c_i 's are positive and hence $C_0 = \Delta^k P(0) \leq 0$. We claim that $C_0 < 0$. Otherwise, by Theorem 1.2 and [17, Theorem 6] one gets $\deg(P) \leq 2k - 2$, which is a contradiction.

A Appendix

Combining [20, Proposition 2.4] and [8, Lemma 3.2] we state the following proposition:

Proposition A.1 *Let Ω be an open set in \mathbb{R}^n . Let $u \in C^{2\sigma+\epsilon}(\Omega) \cap L_\sigma(\mathbb{R}^n)$ for some $\sigma \in (0, 1)$ and $\epsilon > 0$. Then $(-\Delta)^\sigma u$ is continuous in Ω and for every $x \in \Omega$ we have*

$$\begin{aligned} (-\Delta)^\sigma u(x) &= C_{n,\sigma} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy \\ &= -\frac{1}{2} C_{n,\sigma} P.V. \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2\sigma}} dy, \end{aligned} \quad (28)$$

where $C^{2\sigma+\epsilon}(\Omega) := C^{0,2\sigma+\epsilon}(\Omega)$ for $2\sigma + \epsilon \leq 1$ and $C^{2\sigma+\epsilon}(\Omega) = C^{1,2\sigma+\epsilon-1}(\Omega)$ for $2\sigma + \epsilon > 1$ and the constant $C_{n,\sigma}$ is given by

$$C_{n,\sigma} := \left(\int_{\mathbb{R}^n} \frac{1 - \cos x_1}{|x|^{n+2\sigma}} dx \right)^{-1}.$$

The advantage of (28) is that the integral is not singular at the origin for a C^2 function.

Proof of the following lemma can be found in [12].

Lemma A.2 (Fundamental solution) *For $n \geq 3$ odd integer the function*

$$\Phi(x) := \frac{\left(\frac{n-3}{2}\right)!}{2\pi^{\frac{n+1}{2}}} \frac{1}{|x|^{n-1}} = \frac{1}{\gamma_n} (-\Delta)^{\frac{n-1}{2}} \log \frac{1}{|x|}$$

*is a fundamental solution of $(-\Delta)^{\frac{1}{2}}$ in \mathbb{R}^n in the sense that for all $f \in L^1(\mathbb{R}^n)$ we have $\Phi * f \in L_{\frac{1}{2}}(\mathbb{R}^n)$ and for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} (\Phi * f) \varphi dx := \int_{\mathbb{R}^n} (\Phi * f) (-\Delta)^{\frac{1}{2}} \varphi dx = \int_{\mathbb{R}^n} f \varphi dx.$$

Lemma A.3 *Let ℓ be a nonnegative integer. Let v be a smooth function on \mathbb{R}^n such that $D^\alpha v \in L^1_{\frac{1}{2}}(\mathbb{R}^n)$ for every multi-index α with $|\alpha| \leq \ell$. Then*

$$(-\Delta)^{\frac{1}{2}} D^\alpha v(x) = D^\alpha (-\Delta)^{\frac{1}{2}} v(x), \quad \text{for every } x \in \mathbb{R}^n, |\alpha| \leq \ell.$$

Proof. It suffices to show the case for $|\alpha| = 1$. Let $\varphi \in C_c^\infty(B_2)$ be such that $\varphi = 1$ on B_1 and $0 \leq \varphi \leq 1$. Let us define $v_k(x) := \varphi(\frac{x}{k})v(x)$. Then we have

$$(-\Delta)^{\frac{1}{2}} D^\alpha v_k(x) = D^\alpha (-\Delta)^{\frac{1}{2}} v_k(x). \quad (29)$$

We claim that

$$(-\Delta)^{\frac{1}{2}} D^\alpha v_k \xrightarrow{k \rightarrow \infty} (-\Delta)^{\frac{1}{2}} D^\alpha v \quad \text{in } C_{loc}^0(\mathbb{R}^n), \quad |\alpha| = 0, 1.$$

To prove our claim first we fix a $R > 0$. Then for $x \in B_R$ and $k \geq R + 1$ we get

$$\begin{aligned} \left| (-\Delta)^{\frac{1}{2}} D^\alpha v_k(x) - (-\Delta)^{\frac{1}{2}} D^\alpha v(x) \right| &= C_{n, \frac{1}{2}} \left| P.V. \int_{\mathbb{R}^n} \frac{D^\alpha v_k(x) - D^\alpha v_k(y) - D^\alpha v(x) + D^\alpha v(y)}{|x - y|^{n+1}} dy \right| \\ &\leq C_{n, \frac{1}{2}} \int_{|y| > k} \frac{2|D^\alpha v(y)| + |\alpha|k^{-1}|(D^\alpha \varphi)(\frac{y}{k})||v(y)|}{|x - y|^{n+1}} dy \\ &\xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Thus $\{D^\alpha (-\Delta)^{\frac{1}{2}} v_k\}_{k=1}^\infty = \{(-\Delta)^{\frac{1}{2}} D^\alpha v_k\}_{k=1}^\infty$ and $\{(-\Delta)^{\frac{1}{2}} v_k\}_{k=1}^\infty$ are Cauchy sequences in $C_{loc}^0(\mathbb{R}^n)$ and consequently

$$D^\alpha (-\Delta)^{\frac{1}{2}} v_k(x) \xrightarrow{k \rightarrow \infty} D^\alpha (-\Delta)^{\frac{1}{2}} v(x),$$

and together with (29) complete the proof. \square

Lemma A.4 *Let $h \in C^{n-1}(\bar{B}_r)$ be such that*

$$\begin{cases} (-\Delta)^{\frac{n-1}{2}} h = 0 & \text{in } B_r \\ (-\Delta)^j h = f_j & \text{on } \partial B_r, j = 0, 1, \dots, \frac{n-3}{2}. \end{cases} \quad (30)$$

Then for every $x \in B_r$

$$h(x) = - \sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} f_i(y) \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(x, y) \right) d\sigma(y),$$

and

$$|h(x)| \leq C \sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} |f_i(y)| \frac{1}{|x - y|^{n-1-2i}} d\sigma(y), \quad (31)$$

where G is the Green's function corresponding to the problem (30).

Proof. Using integration by parts we have

$$\begin{aligned}
0 &= \int_{B_r} G(x, y) (-\Delta)^{\frac{n-1}{2}} h(y) dy \\
&= \sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} (-\Delta)^i h(y) \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(x, y) \right) d\sigma(y) + \int_{B_r} (-\Delta)^{\frac{n-1}{2}} G(x, y) h(y) dy \\
&= h(x) + \sum_{i=0}^{\frac{n-3}{2}} \int_{\partial B_r} f_i(y) \frac{\partial}{\partial \nu} \left((-\Delta)^{\frac{n-3}{2}-i} G(x, y) \right) d\sigma(y)
\end{aligned}$$

To get (31) we only need to show that

$$\left| \frac{\partial}{\partial y_i} (-\Delta)^j G(x, y) \right| \leq \frac{1}{|x - y|^{2+2j}}, \quad x, y \in B_r, \quad 0 \leq j \leq \frac{n-3}{2}.$$

In order to do that we use the following representation formula of G given by (see e.g. [10])

$$G(x, y) = \underbrace{\int_{B_r} \dots \int_{B_r}}_{\frac{n-3}{2} \text{ times}} G_1(x, z_1) G_1(z_1, z_2) \dots G_1(z_{\frac{n-3}{2}}, y) dz_1 dz_2 \dots dz_{\frac{n-3}{2}}, \quad x, y \in B_r,$$

where

$$G_1(x, y) = \frac{1}{n(n-2)|B_1|} \left(\frac{1}{|x - y|^{n-2}} - \frac{r^{n-2}}{||x|(y - \frac{r^2 x}{|x|^2})|^{n-2}} \right) \quad x, y \in B_r,$$

is the Green's function for Laplacian on B_r . Then for $0 \leq j \leq \frac{n-3}{2}$

$$(-\Delta)^j G(x, y) = \underbrace{\int_{B_r} \dots \int_{B_r}}_{\frac{n-3-2j}{2} \text{ times}} G_1(x, z_1) G_1(z_1, z_2) \dots G_1(z_{\frac{n-3-2j}{2}}, y) dz_1 dz_2 \dots dz_{\frac{n-3-2j}{2}},$$

and

$$\frac{\partial}{\partial y_i} (-\Delta)^j G(x, y) = \underbrace{\int_{B_r} \dots \int_{B_r}}_{\frac{n-3-2j}{2} \text{ times}} G_1(x, z_1) G_1(z_1, z_2) \dots \frac{\partial}{\partial y_i} G_1(z_{\frac{n-3-2j}{2}}, y) dz_1 dz_2 \dots dz_{\frac{n-3-2j}{2}}.$$

A repeated use of Lemma 3.3 and the estimate

$$0 < G_1(x, y) \leq \frac{C}{|x - y|^{n-2}} \quad \text{and} \quad \left| \frac{\partial}{\partial x_i} G_1(x, y) \right| \leq \frac{C}{|x - y|^{n-1}} \quad x, y \in B_r,$$

gives

$$\left| \frac{\partial}{\partial y_i} (-\Delta)^j G(x, y) \right| \leq C \int_{B_r} \frac{1}{|x - z|^{3+2j}} \frac{1}{|y - z|^{n-1}} dz \leq C \frac{1}{|x - y|^{2+2j}}, \quad 0 \leq j \leq \frac{n-3}{2}.$$

□

Lemma A.5 *We set*

$$f_0(x) := \log |x|, \quad f_j(x) := \frac{1}{|x|^j} \text{ for } j = 1, 2, \dots, n-1.$$

Then for $0 < \sigma < 1$ we have

$$(-\Delta)^\sigma f_j(x) = \frac{1}{|x|^{j+2\sigma}} (-\Delta)^\sigma f_j(e_1), \quad \text{for } |x| > 0 \text{ and } 0 \leq j \leq n-1.$$

Proof. Since $f_j \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L_{\frac{1}{2}}(\mathbb{R}^n)$ using (28) we get

$$\begin{aligned} (-\Delta)^\sigma f_j(x) &= (-\Delta)^\sigma f_j(|x|e_1) = c_n P.V. \int_{\mathbb{R}^n} \frac{f_j(|x|e_1) - f_j(y)}{||x|e_1 - y|^{n+2\sigma}} dy \\ &= \frac{1}{|x|^{j+2\sigma}} c_n P.V. \int_{\mathbb{R}^n} \frac{f_j(e_1) - f_j(y)}{|e_1 - y|^{n+2\sigma}} dy \\ &= \frac{1}{|x|^{j+2\sigma}} (-\Delta)^\sigma f_j(e_1), \end{aligned}$$

where in the first equality we used that the function $(-\Delta)^\sigma f_j$ is radially symmetric. \square

The following lemma is a variant of [17, Theorem 6].

Lemma A.6 *Let $v \in L_{\frac{n}{2}}(\mathbb{R}^n)$ and let $h = u - v$ be $\frac{n+1}{2}$ -harmonic in \mathbb{R}^n i.e.*

$$\Delta^{\frac{n+1}{2}} h = 0, \quad \text{in } \mathbb{R}^n.$$

If u satisfies (13) then h is a polynomial of degree at most $n-1$.

Proof. First notice that the condition $v \in L_{\frac{n}{2}}(\mathbb{R}^n)$ implies that

$$\int_{B_R} |v| dx = o(R^{2n}) \quad \text{as } R \rightarrow \infty.$$

For a fixed $x \in \mathbb{R}^n$ by [17, Proposition 4] we have

$$|D^\alpha h(x)| \leq \frac{C}{R^{2n}} \int_{B_R(x)} |h(y)| dy \leq \frac{C}{R^{2n}} \int_{B_{2R}} |h(y)| dy, \quad \alpha \in \mathbb{N}^n \text{ with } |\alpha| = n, \text{ as } R \rightarrow \infty.$$

Now using (13)

$$\int_{B_R} h^+ dx \leq \int_{B_R} (u^+ + |v|) dx = o(R^{2n}) \quad \text{or} \quad \int_{B_R} h^- dx \leq \int_{B_R} (u^- + |v|) dx = o(R^{2n}).$$

On the other hand, Pizzetti's formula (see e.g. [17, Lemma 3]) implies that

$$\int_{B_R} h dx = O(R^{2n-1}), \quad \text{as } R \rightarrow \infty.$$

Therefore,

$$|D^\alpha h(x)| \leq \frac{C}{R^{2n}} \min \left\{ \int_{B_{2R}} (2h^+ - h) dy, \int_{B_{2R}} (2h^- + h) dy \right\} = \frac{1}{R^{2n}} (O(R^{2n-1}) + o(R^{2n})) \\ \xrightarrow{R \rightarrow \infty} 0,$$

and hence h is a polynomial of degree at most $n - 1$. \square

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References

- [1] N. ABATANGELO: *Large s -harmonic functions and boundary blow-up solutions for the fractional Laplacian*, arXiv:1310.3193 (2013).
- [2] H. BRÉZIS, F. MERLE: *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differ. Equ. **16** (1991), 1223-1253.
- [3] C. BUCUR: *Some observations on the Green function for the ball in the fractional Laplace framework*, arXiv:1502.06468 (2015).
- [4] S-Y. A. CHANG; W. CHEN: *A note on a class of higher order conformally covariant equations*, Discrete Contin. Dynam. Systems **63** (2001), 275-281.
- [5] S-Y. A. CHANG, P. C. YANG: *On uniqueness of solutions of n -th order differential equations in conformal geometry*, Math. Res. Lett. **4** (1997), 91-102.
- [6] W. CHEN, C. LI: *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (3) (1991), 615-622.
- [7] F. DA LIO, L. MARTINAZZI, T. RIVIÈRE: *Blow-up Analysis of a nonlocal Liouville-type equation*, arXiv:1503.08701 (2015).
- [8] E. DI NEZZA, G. PALATUCCI, E. VALDINOCI: *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521-573.
- [9] F. GAZZOLA, H-C. GRUNAU, G. SWEERS: *Polyharmonic boundary value problems*, Springer-Verlag, Berlin, 2010. xviii+423 pp. ISBN: 978-3-642-12244-6.
- [10] H.-CH. GRUNAU, G. SWEERS: *Sharp estimates for iterated Green functions*, Proceedings of the Royal Society of Edinburgh, **132A** (2002), 91-120.
- [11] X. HUANG, D. YE: *Conformal metrics in \mathbb{R}^{2m} with constant Q -curvature and arbitrary volume*, arXiv:1504.00565 (2015).
- [12] A. HYDER: *Existence of entire solutions to a fractional Liouville equation in \mathbb{R}^n* , arXiv:1502.02685 (2015).
- [13] A. HYDER, L. MARTINAZZI: *Conformal metrics on \mathbb{R}^{2m} with constant Q -curvature, prescribed volume and asymptotic behavior*, Discrete Contin. Dynam. Systems A **35** (2015), no.1, 283-299.

- [14] T. JIN, A. MAALAOU, L. MARTINAZZI, J. XIONG: *Existence and asymptotics for solutions of a non-local Q -curvature equation in dimension three*, Calc. Var. Partial Differential Equations **52** (2015) no. 3-4, 469-488.
- [15] E.H. LIEB, M. LOSS: *Analysis*, Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001. ISBN:0-8218-2783-9.
- [16] C. S. LIN: *A classification of solutions of a conformally invariant fourth order equation in \mathbb{R}^n* , Comment. Math. Helv. **73** (1998), no. 2, 206-231.
- [17] L. MARTINAZZI: *Classification of solutions to the higher order Liouville's equation on \mathbb{R}^{2m}* , Math. Z. **263** (2009), no. 2, 307-329.
- [18] L. MARTINAZZI: *Conformal metrics on \mathbb{R}^{2m} with constant Q -curvature and large volume*, Ann. Inst. Henri Poincaré (C) **30** (2013), 969-982.
- [19] X. ROS-OTON, J. SERRA: *The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), no. 3, 275302.
- [20] L. SILVESTRE: *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67112.
- [21] J. WEI, D. YE: *Nonradial solutions for a conformally invariant fourth order equation in \mathbb{R}^4* , Calc. Var. Partial Differential Equations **32** (2008), no. 3, 373-386.
- [22] X. XU: *Uniqueness and non-existence theorems for conformally invariant equations*, J. Funct. Anal. **222** (2005), no. 1, 128.